Topological properties of (tall) monotone complexity one spaces

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University of Cologne

Meeting in honour of Michèle Vergne's 80th birthday

06.09.2023

Based on:

- "*On topological properties of positive complexity one spaces*", S. and Sepe, *Transformation Groups* **9** (2020).
- "*Tall and monotone complexity one spaces of dimension six*", Charton, PhD Thesis, Cologne 2021.
- "*Compact monotone tall complexity one T-spaces*" Charton, S. and Sepe, arXiv:2307.04198 [math.SG].

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$$c_1 = \lambda[\omega] \quad (\text{with } \lambda > 0)$$

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Henceforth consider *positive monotone symplectic manifolds*

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(Example: $\dim_{\mathbb{C}}(Y) = 1 \implies Td(Y) = \frac{c_1}{2}[Y]$,

$\dim_{\mathbb{C}}(Y) = 2 \implies Td(Y) = \frac{c_1^2 + c_2}{12}[Y]$,

$\dim_{\mathbb{C}}(Y) = 3 \implies Td(Y) = \frac{c_1 c_2}{24}[Y]$)

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What if one assumes that (M, ω) has symmetries?

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Assume $T \circlearrowleft (M,\omega)$ is *Hamiltonian*:

$\exists\ \psi\colon (M,\omega) \to Lie(T)^*$ (*moment map*) s.t.

- ψ is T-invariant
- $\forall \xi \in Lie(T)$

$$d\langle \psi, \xi \rangle = -\iota_{X_\xi}\omega$$

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$$\psi_2 \circ \Psi = a \circ \psi_1$$

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- Finitely many examples in each dimension? (Modulo equivalence)

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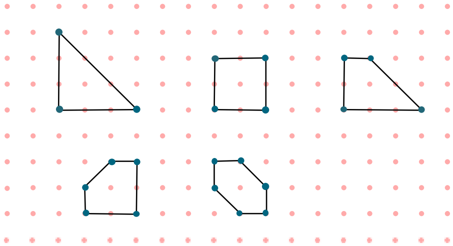
E.g. $\dim(M) = 4$, modulo $GL(2, \mathbb{Z})$, $\psi(M) =$



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Let (M, ω, ψ) be a compact Hamiltonian T-space. For any $\alpha \in \psi(M)$, $\pi_1(M) \simeq \pi_1(M_\alpha)$, where $M_\alpha = \psi^{-1}(\alpha)/T$ is the reduced space at α.

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To prove (b): prove that $\psi^{-1}(v)$ is a **sphere**.

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- $\log(DH)$ is concave (Cho-Kim)

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$$DH\colon \quad \psi(M) \to \mathbb{R}$$

For a Hamiltonian T-space:

- $DH(\alpha)$ = symplectic volume of M_α, $\quad \alpha$ regular (Duistermaat-Heckman)
- There exists a continuous representative of DH.

For complexity one:

- The DH-function is piece-wise linear (Duistermaat-Heckman)
- $\log(DH)$ is concave (Cho-Kim)
  $\Longrightarrow$ DH is **concave**

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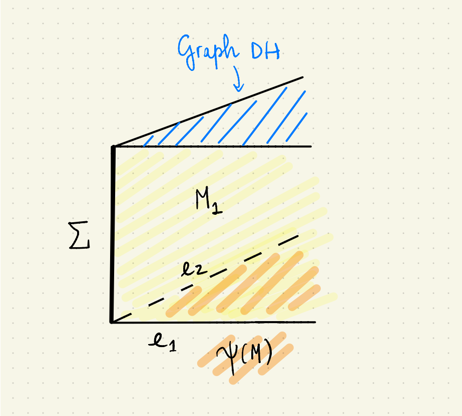
- $N_\Sigma = N_1 \oplus \cdots \oplus N_{n-1}$
- $M_i := \psi^{-1}(e_i)$: compact symplectic 4-dimensional submanifold with a Hamiltonian S^1 action, $\Sigma \subset M_i$, for all $i = 1, \ldots, n-1$
- Normal bundle to Σ in M_i is N_i



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- DH attains its minimum at $v_{\min} \implies$

$$c_1(N_i)[\Sigma] \leq 0 \quad \forall i = 1, \ldots, n-1$$

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Duistermaat-Heckman function

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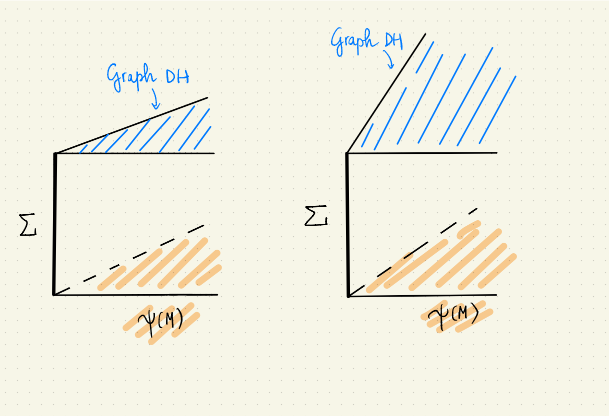

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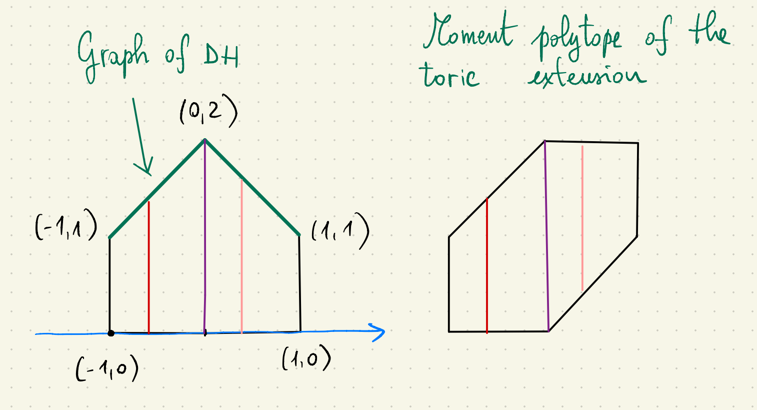
Example:


Graph of DH
(0,2)
(-1,1)
(1,1)
(-1,0)
(1,0)
Moment polytope of the
toric extension

THANK YOU!...

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et joyeux anniversaire!